Related Expressions for Bernoulli's and Euler's Numbers.

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By the formula*

$$\left(\frac{d}{dx}\right)^n \phi(u) = \sum_1^n {}_r \sum_r^n {}_\rho \frac{(-1)^{\rho-r} u^{\rho-r} \phi^\rho(u)}{r!\,(\rho-r)!} \left(\frac{d}{dx}\right)^n u^r, \tag{1}$$

we may very readily connect Bernoulli's and Euler's numbers.

Put $u = e^{-ix}$, then

$$\sec x + \tan x = \frac{2}{u+i} + i = \sum \frac{E_n x^n}{n!}, \tag{2}$$

where, when n is even, E_n is one of Euler's numbers, and when n is odd, Bernoulli's number $B_{\frac{n+1}{2}} = \frac{(n+1)\,E_n}{2^{n+1}(2^{n+1}-1)}$. Putting $\phi(u) = \frac{1}{u+i}$ in (1), we get

$$\left(\frac{d}{dx}\right)^n \frac{1}{u+i} = \sum_1^n {}_r \sum_r^n {}_\rho \frac{(-1)^r \rho!\, u^{\rho-r}(u+i)^{-(\rho+1)}}{r!\,(\rho-r)!} \left(\frac{d}{dx}\right)^n e^{-rix}, \tag{3}$$

whence

$$\frac{1}{2} E_n = \left(\frac{d}{dx}\right)^n \frac{1}{u+i} = \sum_1^n {}_r \sum_r^n {}_\rho \frac{(-1)^r \rho!\,(-ri)^n (1+i)^{-(\rho+1)}}{r!\,(\rho-r)!}$$

$$= \sum_1^n {}_r \sum_r^n {}_\rho \frac{(-1)^r(-ri)^n}{r!} \underset{y=(1+i)^{-1}}{y^{r+1}\left(\frac{d}{dy}\right)^r y^\rho} = \sum_1^n {}_r \frac{(-1)^r(-ri)^n}{r!} y^{r+1}\left(\frac{d}{dy}\right)\frac{1-y^{n+1}}{1-y}$$

$$= \sum_1^n {}_r \frac{(-1)^r(-ri)^n}{r!} y^{r+1} \left\{ \frac{r!}{(1-y)^{r+1}} = \sum_0^r {}_s\, r! \binom{n+1}{s} \frac{y^{n-s+1}}{(1-y)^{r-s+1}} \right\}$$

$$= \sum_1^n {}_r (-1)^r(-ri)^n \left(\frac{y}{1-y}\right)^{r+1} \left\{ 1 - y^{n+1} \sum_0^r {}_s \binom{n+1}{s} \left(\frac{1-y}{y}\right)^s \right\}$$

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*For demonstration of this formula see Bertrand's Calcul Différentiel, p. 141, or *Amer. Journ. of Math.*, Vol. XI, p. 390, formula (7).

$$= y^{n+1}\sum_{1}^{n}{}_{r}(-1)^r(-ri)^n\left(\frac{y}{1-y}\right)^{r+1}\sum_{r+1}^{n+1}{}_{s}\binom{n+1}{s}\left(\frac{1-y}{y}\right)^s$$

$$= y^{n+1}\sum_{2}^{n+1}{}_{s}\sum_{1}^{s-1}{}_{r}\binom{n+1}{s}(-1)^r(-ri)^n\left(\frac{1-y}{y}\right)^{s-r-1}$$

$$= \left(-\frac{1+i}{2}\right)^{n+1}\sum_{2}^{n+1}{}_{s}\sum_{1}^{s-1}{}_{r}i^s\binom{n+1}{s}\,.\,i^r r^n.$$

Observing that $(1+i)^2 = 2i$, we find

$$\left.\begin{aligned} E_n &= 2\left(\frac{i}{2}\right)^{\frac{n+1}{2}}\sum_{2}^{n+1}{}_{s}\sum_{1}^{s-1}{}_{r}i^{s+r}\binom{n+1}{s}r^n, \quad (n\text{ odd}) \\ E_n &= -(1+i)\left(\frac{i}{2}\right)^{\frac{n}{2}}\sum_{2}^{n+1}{}_{s}\sum_{1}^{s-1}{}_{r}i^{s+r}\binom{n+1}{s}r^n, \quad (n\text{ even}). \end{aligned}\right\} \tag{4}$$

When n is odd we have then

$$B_{\frac{n+1}{2}} = \frac{2(n+1)}{2^{n+1}(2^{n+1}-1)}\left(\frac{i}{2}\right)^{\frac{n+1}{2}}\sum_{2}^{n+1}{}_{s}\sum_{1}^{s-1}{}_{r}i^{s+r}\binom{n+1}{s}r^n.$$

Taking account only of the real terms in the summations on the right side of (4), since E_n being real, the imaginary terms must cancel one another, we may also write (4) under the form

$$\left.\begin{aligned} E_n &= (-1)^{\left[\frac{n}{4}\right]+(-1)^{\frac{n+1}{2}}}2^{-\frac{n+1}{2}} \\ &\quad \sum_{2}^{n+1}{}_{s}\binom{n+1}{s}\left\{\left(1-(-1)^{\frac{n+1}{2}}\right)v_{s-1}-\left(1+(-1)^{\frac{n+1}{2}}\right)v_{s-2}\right\}, \quad (n\text{ odd}) \\ E_n &= (-1)^{\left[\frac{n}{4}\right]}2^{-\frac{n}{2}}\sum_{2}^{n+1}{}_{s}\binom{n+1}{s}\left\{v_{s-1}+(-1)^{\frac{n}{2}}v_{s-2}\right\}, \quad (n\text{ even}) \end{aligned}\right\} \tag{5}$$

where $(-1)^s v_s = s^n - (s-2)^n + (s-4)^n - \ldots.$